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On Difference Orlicz Space χ^{π}

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Abstract **-** This paper is devoted to a study of the general properties of χ_M^{π} in respect of the difference sequence space $\chi_M^{\pi}(\Delta)$.
Keywords: χ-sequence, difference sequence, analytic sequence, Orlicz space

INTRODUCTION

A complex sequence, whose k^{th} term is x_k is denoted by $\{x_k\}$ or simply x. Let ϕ be the set of all finite sequences. A sequences $x = \{x_k\}$ is said to be analytic if $\sup_k \left(|x_k|\right)^{\frac{1}{k}} < \infty$. The vector space of all analytic sequences will be denoted by Λ. A sequence $x = \{x_k\}$ is said to be entire sequences if $\lim_{k\to\infty}\left(|x_k|\right)^{\frac{1}{k}} = 0$. The vector space of all entire sequences will be denoted by Γ. A sequence x is called gai sequence if $\lim_{k\to\infty}\left(k!|x_k|\right)^{\frac{1}{k}} = 0$. The vector space of all gai of phi sequence will be denoted by χ^{π}. kizmaz [33] defined the following difference sequence spaces

$$Z(\Delta) = \{x = (x_k) : \Delta_x \in Z\}$$

for $Z = l_\infty, c, c_0$, where $\Delta_x = (\Delta x)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$ and showed that these are banach spaces with norm $\|x\| = |x_1| + \|\Delta_x\|_\infty$. Later on Et and colak [15] generalized the notion as follows:

Let $m\in N$, $Z(\Delta^m) = \{x = (x_k) : \Delta^m x \in Z\}$ for $Z = l_\infty$, c, c_0 where $m\in N$.

$$\Delta^0 x = (x_k),\ (\Delta_x) = (x_k - x_{k+1}),$$

$$\Delta^m x = (\Delta^m x_k)_{k=1}^{\infty} = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})_{k=1}^{\infty}$$

The generalized difference has the following binomial representation :

$$\Delta^m x_k = \sum_{\gamma=0}^{m} (-1)^{\gamma} \binom{m}{\gamma} x_{k+\gamma},$$

They proved that these are Banach spaces with the norm $\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$

Orlicz [19] used the idea of Orlicz function to construct the space (L^M). Lindenstranss and Tzafriri [1] investigated Orlicz sequence space in more detail, and they proved that every Orlicz sequence space l_M contains a subspace isomorphic to l_p $(1 \le p < \infty)$ subsequently, different classes of sequence spaces were defined by Parashar and Choudhry [2], Mursaleenetel [3], Bektas and Altin [4], Tripathy et at. [5], Rao and subramanian [6], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [7].

An Orlicz function is a function M : $[0,\infty)$ which is continuous, non-decreasing and convex with M(0) = 0, M(x) > 0, x > 0 and M(x) $\to\infty$ as $x\to\infty$. If the convexity of Orlicz function M is replaced by M $(x + y) \le$ M(x) + M(y). then this function is called modulus function, introduced by Nakano [18] and further discussed by Ruckle [8] and Maddox [9], and many others.

An Orlicz function M is said to satisfy Δ2 condition for all values of u, if there exists a constant K > 0, such that M (2u) ≤ K M(u) (u≥0). The condition Δ_2 is equivalent to M $(lu) \le$ K l M(u), for all values of u and for $l > 1$. Lindestrauss and Tzafriri [1] used the idea of Orlicz function to construct Orlicz sequence space

$$l_M = \left\{x\in\omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\ some\ \rho > 0\right\} \quad ...(1)$$

The space l_M with the norm

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < 1\right\} \quad ...(2)$$

becomes a Banach space which is called an Orlicz sequence space. For M(t),t^p , $(1 \le p < \infty)$ the space l_M coincide with the classical sequence space l_p. For $(0 \le r \le 1)$, a non-void subset U of a linear space is said to be absolutely γ-convex if $x,y\in$ U and $|\lambda|^r + |\mu|^r \le 1$ together imply $\lambda_x + \mu_y \in$ U for $\lambda,\mu\in$ C. It is clear that if U is absolutely γ-convex, then it is absolutely t-convex for $t<r$. A linear topological space X is said to be r-convex neighbourhood of 0 $\in X$. The r-convexity for r > 1 is of little interest, since X is r-convex for r > 1 if and only if X is the only neighbourhood of 0 ∈ X .

Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence

$x^n = \{x_1, x_2, \ldots\ldots, x_n, 0, 0, \ldots.\}$

$s^{(k)} = (0, 0, \ldots., 1/k!, 0, 0, \ldots.\}$,, (1/k!) in the k^{th} place and zero's else where

If X is a sequence space, we define

(i) X′ = the continuous dual of X

(ii) $X^{\alpha} = \{a = (a_k) : \sum_{K=1}^{\infty} |a_k x_k| < \infty, \text{ for each } x \in X\}$

(iii) $X^{\beta} = (a = (a_k) : \sum_{K=1}^{\infty} a_k x_k$ is convergent, for each $x \in X\}$

(iv) $X^{\gamma} = \{a = (a_k : \sup_n \left|\sum_{K=1}^{\infty} a_k x_k\right| < \infty$, for each $x \in X\}$

(V) Let X be an FK space ⊃ ϕ.

Then $X^f = \{f(\delta^n) : f \in X'\}$

X^{α}, X^{β}, X^{γ} are called the α - (or KÖ the T öeplitz) dual of X. β-(or generalized K ö the - T öeplitz) dual of X, γ-dual of X. Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for μ = α, β, or γ.

An FK space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals $p_k(x) = x_k$(k = 1,2,3...) are continuous.

An FK - space is a locally convex Frechet space which is made up of sequences and has the property that coordinate projections are continuous. A metric space (X,d) is said to have AK (or sectional convergence) if and only if $d(x^n, x) \to 0$ as $n \to \infty$. [10] The space is said to have AD (or) be an AD space if ϕ is dense in X. We note that implies AK implies AD by [4].

2. DEFINITION AND PRELIMINARIES

Throughout this paper ω, χ_M^{π}, Γ_M and Λ_M denote the spaces of all, Orlicz space of gai sequences, Orlicz space of entire sequences and Orlicz space of bounded sequences respectively. In this paper we define and study the orlicz difference sequence spaces of gai sequences, entire sequences and analytic sequences. The idea of difference sequences was first introduced by Kizmaz [33] write $\Delta x_k = x_k - x_{k+1}$, for $k = 1,2,3, \ldots.$

Let ω denote the set of all complex sequences $x = \{x_k\}_{k=1}^{\infty}$, Δ : ω→ω be the difference operator defined by $\Delta x = (x_k - x_{k+1})_{k=1}^{\infty}$, and M : [0,∞) → [0,∞) be an orlicz function. Define the sets

$$\chi_M^{\pi} = \left\{ x \in \omega : \left(M\left(\frac{\left(k!|x_k|\right)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho} \right) \right) \to 0 as k \to \infty for some \rho > 0 \right\}$$

$$\Gamma_M = \left\{ x \in \omega : \left(M\left(\frac{\left(|x_k|\right)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho} \right) \right) \to 0 as k \to \infty for some \rho > 0 \right\}$$

$$\Lambda_M = \left\{ x \in \omega : \sup_k \left(M\left(\frac{\left(|x_k|\right)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho} \right) \right) < \infty for some \rho > 0 \right\}$$

Define the sets

$\chi_M^{\pi}(\Delta) = \{x \in \omega : \Delta_x \in \chi_M^{\pi}\}$;

$\Gamma_M(\Delta) = \{x \in \omega : \Delta_x \in \Gamma_M\}$;

$\Lambda_M(\Delta) = \{x \in \omega : \Delta_x \in \Lambda_M\}$;

The space $\chi_M^{\pi}(\Delta)$ is a metric space with the metric

$$d(x,y) = \inf\left\{ \rho > o : \sup_k \left(M\left(\frac{(k!|\Delta x_k - \Delta y_k)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho} \right) \right) \leq 1 \right\}$$

for all $x = \{x_k\}$ and $y = \{y_k\}$ in $\chi_M^{\pi}(\Delta)$. The space $\Lambda_M(\Delta)$, $\Gamma_M(\Delta)$

2.1 Definition

A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ when ever $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$.

2.2 Lemma

Let X be an FK-space ⊃ ϕ. Then

(i) $X^{\gamma} \subset X^f$

(ii) if X has AK, $X^{\beta} = X^f$

(iii) if X has AD, $X^{\beta} = X^{\gamma}$.

3. MAIN RESULTS

3.1 Proposition

$(\chi_M^\pi(\Delta) \subset \Gamma_M(\Delta))$

Proof : Proof is easy, so omitted.

3.2 Proposition

$\chi_M^\pi(\Delta)$ has AK

Proof

Let $x \in \chi_M^\pi(\Delta)$, so $\{\Delta x_k\} \in \chi_M^\pi$.

Then $\lim_{k\to\infty}\left(M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right)=0$ and hence

$$\sup_{k\geq n+1}\left(M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right)=0 \text{ as } n\to\infty \qquad ...(3)$$

$$d(x,x^{[n]})=\inf\left\{\rho>0:\sup_{k\geq n+1}\left(M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right)\leq 1\right\}\to 0$$

as $n\to\infty \Rightarrow x^{[n]}\to x$ as $n\to\infty$ implying that $\chi_M^\pi(\Delta)$ has AK.

This completes the proof.

3.3 Proposition

$\chi_M^\pi(\Delta)$ is not solid.

3.3.1 Example

consider $(x_k) = (1) \in \chi_M^\pi(\Delta)$. Let $\alpha_k = (-1)^k$, then $(\alpha_k x_k)$ not $\in \chi_M^\pi(\Delta)$. Hence $\chi_M^\pi(\Delta)$ is not solid.

3.4 Proposition

Let M be an Orlicz function which satisfies Δ_2 condition. Then $\chi^\pi(\Delta) \subset \chi_M^\pi(\Delta)$

Proof :

Let $x\in\chi^\pi(\Delta)$ .... (4)

Then $(K!|\Delta xk|)^{\frac{1}{k}} \leq \in$ for sufficiently large k and every $\in > 0$. But then, by taking $\rho \geq 1/2$.

$$\left\{M\left(\frac{(K!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right\}\leq\left(M\left(\frac{\in}{\rho}\right)\right)\leq(M(2\in))$$ (because M is non-decreasing)

$$\Rightarrow \left\{M\left(\frac{(K!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right\}\leq K(M(\in))$$ *by* Δ_2 *- condition, for some* $K>0$.

$$\Rightarrow \left\{M\left(\frac{(K!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right\}\leq \in \to 0 \text{ as } k\to\infty$$ by defining M $(\in) < \frac{\in}{K}$.

Hence $x\in\chi_M^\pi(\Delta)$ ....(5)

Hence (4) and (5) we get $\chi^\pi(\Delta)\subset\chi_M^\pi(\Delta)$.

This completes the proof.

3.5 Proposition

If M is a Orlicz function, then $\chi_M^\pi(\Delta)$ is a linear space over the set of complex number C.

Proof:

Let $x, y\in\chi_M^\pi(\Delta)$ and $\alpha, \beta \in$ C.

Then there exist positive real numbers ρ_1 and ρ_2 such that

$$M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho_1}\right)\to 0 \text{ as } k\to\infty \text{ by (6)}$$

$$M\left(\frac{(k!|\Delta y_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho_2}\right)\to 0 \text{ as } k\to\infty \text{ by (7)}$$

Let $\rho_3 = \max\{2A\rho 1, 2B\rho 2\}$ where $A=\sup_k|\alpha|^{\frac{1}{k}}$ $B=\sup_k|\beta|^{\frac{1}{k}}$

Since M is a non decreasing modulus function , we have

$$M\left(\frac{(k!|\alpha\Delta x_k+\beta\Delta y_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho_3}\right)\leq \frac{1}{2}\left(M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho_1}\right)+\frac{1}{2}\left(M\left(\frac{(k!|\Delta y_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho_2}\right)\to 0 \text{ as } k\to\infty \text{ by (6) and (7)}$$

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So $(\alpha x+\beta y)\in\chi_M^{\pi}(\Delta)$

This completes the proof.

3.6 Definition

Let $\rho = \{p_k\}$ be any sequence of positive real numbers. Then we define

$$\chi_M^{\pi}(\Delta,p)=\left\{x=\{x_k\}:\left(M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho_1}\right)\right)^{p_k}\to 0 \text{ as } k\to\infty\right\}$$

when $p_k = p$, a constant for all $k\in N$, then

$\chi_M^{\pi}(\Delta,p)=\chi_M^{\pi}(\Delta)$. The following result can be proved by using standard techniques, so we state the result with out proof.

3.7 Theorem

(a) Let $0<\inf_{p_k}\le p_k\le 1. Then\ \chi_M^{\pi}(\Delta,p)\subset\chi_M^{\pi}(\Delta)$

(b) Let $0\le p_k\le\sup_{p_k}\le\infty. Then\ \chi_M^{\pi}(\Delta)\subset\chi_M^{\pi}(\Delta,p)$

(c) Let $0<p_k\le q_k$ and let $\left\{\frac{q_k}{p_k}\right\}$ be bounded.

Then $\chi_M^{\pi}(\Delta,q)\subset\chi_M^{\pi}(\Delta,p)$

3.8 Theorem

$\chi_M^{\pi}(\Delta)$ is a r convex for all r > 0, where $0\le r\le\inf_{p_k}$. Moreover if $p_k = p\le 1$ for all $k\in N$, then $\chi_M^{\pi}(\Delta,p)$ is p-convex.

Proof : Let $x\in\chi_M^{\pi}(\Delta,p)$. But if $r\in(0, \inf p_k)$ then clearly $r<p_k$ for all k. Let $g^*(x)$ define under the metric

$$g^*(x)=\inf\left\{\rho>0:\sup_k\left(M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right)^{pk}\le 1\right\}$$

Since $r\le p_k\le 1$, for all $k>k_0$.

$g^*(x)$ is sub additive. Further for $0\le|\lambda|\le 1. |\lambda|^{pk}\le|\lambda|^r$, $for\ all\ k>k_0$

Therefore, for each λ we have $g^*(\lambda x)\le|\lambda|^r.g^*(x)$.

Now, for $0<\delta<1$, $U=\{x: g^*(x)\le\delta\}$

which is an absolutely r-convex set, for $|\lambda|^r+|\mu|^r\le 1$ and $x,y\in U$, Now $g^*(\lambda x+\mu y)\le g^*(\lambda x)+g^*(\mu y)$

$\le\ |\lambda|^r g^*(x)+|\mu|^r g^*(y)$

$\le\ |\lambda|^r\delta+|\mu|^r\delta$

$\le\ (|\lambda|^r+|\mu|^r)\delta$

$\le\ \delta$

If $p_k = p\le 1$for all $k\in N$ then $U=\{x: g^*(x)\le\delta\}$, is an absolutely p-convex set. This can be obtained by a similar analysis and therefore we omit the details. This completes the proof.

3.9 Theorem

$(\chi_M^{\pi}(\Delta))^{\beta}=\Lambda$

Proof: step 1 : $\chi_M^{\pi}(\Delta)\subset\Gamma_M(\Delta)$ by Proposition 7.5; then $(\Gamma_M(\Delta))^{\beta}\subset(\chi_M^{\pi}(\Delta))^{\beta}$. But we have $(\Gamma_M(\Delta))^{\beta}=\Lambda$

$\Lambda\subset(\chi_M^{\pi}(\Delta))^{\beta}$ ....(8)

Step 2 : Let $y\in(\chi_M^{\pi}(\Delta))^{\beta}$; $f(x)=\sum_{k=1}^{\infty}x_k y_k$ with $x\in\chi_M^{\pi}(\Delta)$ We recall that $s^{(k)}$ has $1/k!$ in the kth place and zero's elsewhere, with $x = s^{(k)}$.

$$\left\{M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right\}=\left\{0,0,....,M\left(\frac{(1)^{\frac{1}{k}}}{(\pi k)^{\frac{1}{k}}\rho}\right),0,.....\right\}$$

which converges to zero.

Hence $s^{(k)}\in\chi_M^{\pi}(\Delta)$.

Hence $d(s^{(k)},0)=1$

But $|y_k|\le\|f\|d(s^{(k)},0)<\infty$ for all k. Thus (y_k) is a bounded sequence and hence an analytic sequence. In other words $y\in\Lambda$.

$(\chi_M^{\pi}(\Delta))^{\beta}\subset\Lambda$ .... (9)

From (8) and (9) we obtain $(\chi_M^{\pi}(\Delta))^{\beta}=\Lambda$.

This completes the proof.

3.10 Proposition

$$(\chi_M^\pi(\Delta))^\mu = \Lambda for \mu = \alpha,\beta,\gamma,f$$

Proof:

Step 1:

$\left(\chi_M^\pi(\Delta)\right)$ has AK by proposition .

(i) we get $\chi_M^\pi(\Delta)^\beta = \left(\chi_M^\pi(\Delta)\right)^f$. But $\left(\chi_M^\pi(\Delta)\right)^\beta = \Lambda$

Hence $\left(\chi_M^\pi(\Delta)\right)^f = \Lambda$ .... (10)

Step 2 : Since AK implies AD. Hence by Lemma (ii) we get $\left(\chi_M^\pi(\Delta)\right)^\beta = \left(\chi_M^\pi(\Delta)\right)^\Gamma$. Therefore $\left(\chi_M^\pi(\Delta)\right)^\gamma = \Lambda$ ...(11)

Step3 : $\left(\chi_M^\pi(\Delta)\right)$ is not normal by Proposition 2.7. Hence by Proposition 2.7. we get

$\left(\left(\chi_M^\pi(\Delta)\right)^\alpha \neq \left(\chi_M^\pi(\Delta)\right)^\beta\right) = \Lambda$ .... (12) From (10), (11) and (12) we have

$$((\chi_M^\pi(\Delta))^\alpha \neq (\chi_M^\pi(\Delta))^\beta) =$$
$$(\chi_M^\pi(\Delta))^\gamma = (\chi_M^\pi(\Delta))^f = \Lambda$$

3.11 Proposition

The continuous dual space of $\left(\chi_M^\pi(\Delta)\right)$ is Λ. In other words $\left[\chi_M^\pi(\Delta)\right]^* = \Lambda$

Proof : we recall that s^k has $\left(\frac{1}{k!}\right)$ in the k the place zero's elsewhere with $x = s^k$

$$\left\{\frac{M\left(k!\left|\Delta x_k\right|\right)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right\} = ,\left\{\frac{M(1!0)^{\frac{1}{1}}}{(\pi_k)^{\frac{1}{k}}\rho}\right\},\left\{\frac{M(2!0)^{\frac{1}{2}}}{(\pi k)^{\frac{1}{k}}\rho}\right\}$$

$$\left\{\frac{M\left((k-1)!0\right)^{\frac{1}{k-1}}}{(\pi_k)^{\frac{1}{k}}\rho}\right\},\left\{\frac{M(k!(1-0))^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho}\right\},\left\{\frac{M((k+1)!0)^{\frac{1}{k+1}}}{(\pi_k)^{\frac{1}{k}}\rho}\right\}....$$

$$= \left\{0,0,....,\frac{M(1)\frac{1}{k}}{(\pi k)^{\frac{1}{k}}\rho},0,.....\right\}$$

Hence $s^k \in \chi_M^\pi(\Delta)$. We have $f(x) = \sum_{k=1}^{\infty}\Delta x_k y_k$ with $\Delta_x \in \chi_M^\pi(\Delta)$ and $f \in [\chi_M^\pi(\Delta)]^*$, where $[\chi_M^\pi(\Delta)]^*$ is the continuous dual space of $\chi_M^\pi(\Delta)$. Take $x = s^k \in \chi_M^\pi(\Delta)$. Then

$|y_k| \le \|f\| d(s^k,0) < \infty$ for all k

Thus (y_k), is a bounded sequence and hence an analytic sequence.

In other work, y ∈Λ. Therefore $[X_M^\pi(\Delta)]^* = \Lambda$.

3.12 Proposition

$\chi_M^\pi(\Delta)$ is a complete metric space under the metric

$$d(x,y) = \inf\{\rho > 0: \sup_k (M(\frac{(k!|\Delta x_k - \Delta y_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}}\rho} \le 1\}$$

where $x = (x_k) \in \chi_M^\pi(\Delta)$ and $y = (y_k) \in \chi_M^\pi(\Delta)$

Proof:

Let $\{x^{(n)}\}$ be a cauchy sequence in $\chi_M^\pi(\Delta)$

Then given any ∈> 0 there exists a positve integer N depending on ∈.

such that $d(x^{(n)}, x^{(m)}) < \in$, for all $n \ge N$. Hence

$d(x,y) = \inf\left\{\rho > o : \sup_k\left(M\left(\frac{(k!|\Delta x_k - \Delta yk|)^{\frac{1}{k}}}{(\pi k)^{\frac{1}{k}}\rho}\right)\right) \le 1\right\} < \in$ for all $n \ge N$ and for all $m \ge N$

consequently $M\left(\frac{\left(k!\left|\Delta x_k^{(n)}\right|\right)}{(\pi_k)^{\frac{1}{k}}\rho}\right)$ is a cauchy sequence in the metric C of complex numbers. But C is complete. So,

$$\left\{M\left(\frac{\left(k!\left|\Delta x_k^{(n)}\right|\right)}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right\} \to \left\{M\left(\frac{\left(k!\left|\Delta x_k\right|\right)}{(\pi_k)^{\frac{1}{k}}\rho}\right)\right\}, \quad \text{as} \quad n \to \infty.$$

Hence there exist a positive integer n_0 such that

$$d(x,y) = \inf\left\{\rho > o : \sup_k\left(M\left(\frac{(k!|\Delta x_k - \Delta yk|)^{\frac{1}{k}}}{(\pi k)^{\frac{1}{k}}\rho}\right)\right) \le 1\right\} < \in$$

for all n $\geq n_0$. In particular, we have $\left\{ M\left(\frac{(k!|\Delta x_k^{(n)} - \Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) \right\} < \in$. Now

$$\left\{ M\left(\frac{(k!|\Delta xk|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) \right\} \leq M\left(\frac{(k!|\Delta x_k - \Delta x_k^{(n_0)}|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) + \left\{ M\left(\frac{(k!|\Delta x_k^{(n_0)}|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) \right\} < \in + 0 \, as \, k \to \infty.$$

Thus $M\left(\frac{(k!|\Delta x_k|)^{\frac{1}{k}}}{(\pi_k)^{\frac{1}{k}} \rho} \right) < \in as \, k \to \infty$

That is $(x_k) \in \chi_M^\pi(\Delta)$. Therefore, $\chi_M^\pi(\Delta)$ is a complete metric space.

This completes the proof.

3.13 Lemma

$Y \supset X \Leftrightarrow Y^f \subset X^f$ where X is an AD-space and Y be an FK-space.

3.14 Proposition

Let Y be any FK-space $\supset \phi$. Then $Y \supset \chi_M^\pi(\Delta)$ if and only if the sequence $s^{(k)}$ is weakly analytic.

Proof :

The following implications establish the result.

$Y \supset \chi_M^\pi(\Delta) \Leftrightarrow Y^f \subset (\chi_M^\pi(\Delta))^f$, since $\chi_M^\pi(\Delta)$ has AD by lemma 7.17.

$\Leftrightarrow$ $Y^f \subset \Lambda$, since $\left(\chi_M^\pi(\Delta)\right)^f = \Lambda$

$\Leftrightarrow$ for each f $\in$Y′, the topological dual of Y, $f(s^{(k)}) \in \Lambda$

$\Leftrightarrow$ $f(s^{(k)})$ is analytic

$\Leftrightarrow$ $s^{(k)}$ is weakly analytic

This completes the proof.

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